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NOTE

THE JOHNSON GRAPHS SATISFY A DISTANCE EXTENSION PROPERTY

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A graph G has property I_k if whenever F and H are connected graphs with $|F| \le k$ and $|H| = |F| + 1$, and $i : F \to G$ and $j : F \to H$ are isometric embeddings, then there is an isometric embedding $k : H \to G$ such that $k \circ j = i$. It is easy to construct an infinite graph with I_k for all k, and I_2 holds in almost all finite graphs. Prior to this work, it was not known whether there exist any finite graphs with I_3. We show that the Johnson graphs $J(n,3)$ satisfy I_3 whenever $n \ge 6$, and that $J(6,3)$ is the smallest graph satisfying I_3. We also construct finite graphs satisfying I_3 and local versions of the extension axioms studied in connection with the Rado universal graph.

1. Introduction

A graph G has **property** P_k if whenever $x_1, \dots x_i; y_1, \dots, y_j$ is a list of distinct points with $i + j \le k$, then there is some point $z \in G$ not on the list, which is a neighbor of all the x's and none of the $y's$. This kind of condition is called an *extension axiom*; it says that the subgraph induced by some given points may always be extended to one more point in a prescribed way.

The extension axioms P_k originate with Rado's discovery [4] of the *universal countable graph* U. U can be characterized up to isomorphism as the unique countable graph with P_k for all k.

For us, one important fact about the extension axioms is that for fixed k, P_k is satisfied in some *finite* graph. Indeed, if we consider graphs on n points with edges drawn randomly with probability $1/2$, then as $n \to \infty$, P_k holds almost surely. This is well-known from the theory of random graphs.

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Another way to express P_k is in terms of *isomorphic embeddings*, injective maps between graphs preserving the edge relation in both directions. Then P_k can be recast as follows: G has P_k if whenever F and H are graphs with $|F| \leq k$ and $|H| = |F|+1$, and $i : F \to G$ and $j : F \to H$ are isomorphic embeddings, then there is an isomorphic embedding $k : H \to G$ such that $k \circ j = i$.

We are interested here in the following extension axioms for *distances* in graphs: G has **property** I_k if whenever F and H are connected graphs with $|F| \leq k$ and $|H| = |F|+1$, and $i : F \to G$ and $j : F \to H$ are isometric embeddings, then there is an isometric embedding $k : H \to G$ such that $k \circ j = i$.

It is easy to construct an infinite connected graph U_∞ with I_k for all k: in brief, one amalgamates together copies of all the finite connected graphs in all possible ways. U_∞ is a countable connected graph which isometrically embeds every countable connected graph. Note that U_∞ is not the same as the universal graph U mentioned above, since U has diameter 2 and U_∞ has infinite diameter. U_∞ was first studied by Pach [3], and it was rediscovered later in [2].

It was conjectured in [2] that for each k there is a *finite* graph which satisfies I_k. I_2 follows from P_2 and thus holds in almost all finite graphs. But I_3 is a different story; it implies that the diameter of G is ≥ 3 and so contradicts P_2. Up to now, no finite graphs satisfying I_3 had been known.

Here are some consequences of I_3: Let xyz be a triangle. Then there is some w_1 incident to x but neither y nor z, and some w_2 incident to y but neither x nor z, etc.; there is also w_4 incident to x and y but not z, etc. (However, it does not follow that there is some w^* incident to none of $\{x, y, z\}$. This is due to our restriction to connected graphs in the statement of I_k.) For more consequences, consider a every 3-chain $x \sim y \sim z$. For this, we have some v_1 incident to x alone *and of distance 3 from z.* (Again, this consequence of I_3 contradicts P_2.) We also have v_2 incident to x and y but not z, etc. Even this does not exhaust the consequences: every single point must extend to all connected graphs on four points in all possible ways, as must each edge $x \sim y$.

The point of this discussion is that I_3 has many interesting consequences. But as we mentioned earlier, prior to this note we did not know of any finite graphs with I_3. Our main result is that the Johnson graphs $J(n, 3)$ for $n \geq 6$ satisfy I_3. We also show that the smallest graph with I_3 is $J(6, 3)$.

We also have a construction of finite graphs with I_3 which gives graphs that also satisfy another property. We say that G has **property** LP_k if for all $w \in G$, the subgraph $G(w)$ induced by the neighbors of w in G has property P_k. This is a *local* version of property P_k. The graphs U and U_∞ mentioned above have LP_k for all k. For each fixed k, we construct finite graphs satisfying I_3 and P_k. The construction is probabilistic, but it also uses an idea from the work on the Johnson graphs.

2. The Johnson graphs $J(n,3)$ have property I_3

Proposition 2.1. *The following are equivalent:*

1. *G satisfies* I_3.
2. *G is non-empty; G satisfies* LP_2*; every 3-chain $u \sim v \sim w$ can be extended isometrically to a 4-chain $u \sim v \sim w \sim x$; and every 3-chain $u \sim v \sim w$ can be extended to a square $u \sim v \sim w \sim x \sim u$.*

Proof. (1) $\Rightarrow$ (2) is easy. For the converse, we consider various instances of I_3. The assumption that $G \neq \emptyset$ takes care of the case $|F|=0$. LP_2 takes care of $|F|=1$ and $|F|=2$. For $|F|=3$, we have a few subcases. If F is a triangle, then all possibilities for H are accounted for by LP_2. If F is a 3-chain and H is either a 4-chain or a square, we are done by hypothesis. If F is a 3-chain and H is one of the other connected graphs on 4 points, then the extension exists by LP_2. ∎

We shall need to recall two classes of graphs. The *Johnson graphs* $J(n,m)$ have as vertices the m-subsets of an n-set, and $S \sim T$ in $J(n,m)$ if $|S \cap T| = m-1$. Second, the $p \times q$ *rectangle* is the graph $R(p,q)$ whose vertex set is $\{1,\ldots,p\} \times \{1,\ldots,q\}$; $(i,j) \sim (k,l)$ iff either $i=k$ or $j=l$ (but not both). For each point v of $J(n,m)$, the graph $J(n,m)(v)$ induced by the neighbors of v is isomorphic to $R(m,n-m)$. For example, when $v=\{1,2,\ldots,m\}$, the isomorphism is

$$(a,b) \quad \mapsto \quad \{1,2,\ldots,a-1,m+b,a+1,\ldots,m-1,m\}. \tag{1}$$

Theorem 2.1. *For $n \geq 6$, $J(n,3)$ has property* I_3. *In fact, if $n \geq 6$ and $3 \leq m \leq n-3$, then $J(n,m)$ has property* I_3.

Proof. We use Proposition 2.1. As we mentioned just above, $J(n,m)$ is locally isomorphic to $R(m,n-m)$. Assuming that both m and $n-m$ are at least 3, $R(m,n-m)$ easily has P_2. So $J(n,m)$ has LP_2.

Each 3-chain in $J(m,n)$ is of the form

$$\{a_1,\ldots,a_m\} \sim \{a_2,\ldots,a_m,a_{m+1}\} \sim \{a_3,\ldots,a_m,a_{m+1},a_{m+2}\},$$

where all the a's are distinct. Let $b \in \{1,\dots,n\} \setminus \{a_1,\dots,a_{m+2}\}$. We can extend the given 3-chain isometrically to a 4-chain by considering

$$\{a_4,\dots,a_{m+2},b\}.$$

And we can extend the given 3-chain to a square by considering

$$\{a_1,a_3,a_4,\dots,a_m,a_{m+2}\}.$$

∎

Since the diameter of $J(n,m)$ is $min(m,n-m)$, this result also shows that there are graphs of arbitrarily large diameter which satisfy I_3.

2.1. The smallest graph with I_3

Proposition 2.2. *If $|G| \leq 9$ and G has property P_2, then G is isomorphic to $R(3,3)$.*

The proof is elementary (but longer than any of the proofs in this note).

Theorem 2.2. *If $|G| \leq 20$ and G has property I_3, then G is isomorphic to $J(6,3)$.*

Proof. Fix a point x of G. The induced subgraph $G(x)$ of neighbors of x has property P_2. By Proposition 2.2, $deg(x) \geq 9$. By I_3, there is some $\overline{x}$ such that $d(x,\overline{x}) = 3$. Since $deg(\overline{x}) \geq 9$, and since x and $\overline{x}$ have no neighbors in common, we see that $|G|$ must be exactly 20. In addition, $\overline{x}$ is unique. And we have the following *quadrachotomy law*: For all x and y, exactly one of the following four alternatives holds:

$$x = y \quad \text{or} \quad x \sim y \quad \text{or} \quad x \sim \overline{y} \quad \text{or} \quad x = \overline{y}. \tag{2}$$

The uniqueness of $\overline{x}$ implies that $\overline{\overline{x}} = x$. It is not hard to check at this point that the conjugation operation $x \mapsto \overline{x}$ is an automorphism of G.

We define a map $i : G \to J(6,3)$ in three stages. Take any point w of G and set $i_0(w) = \{1,2,3\}$. As we know from Proposition 2.2, $G(w)$ is isomorphic to $R(3,3)$. Using (1) with $m = 3$ and $n = 6$, we can extend i_0 to a map

$$i_1 : \{w\} \cup G(w) \to \{\{1,2,3\}\} \cup J(6,3)(\{1,2,3\}).$$

This map i_1 is a bijection that preserves the edge relation in both directions. Each of the 10 remaining points of G is a conjugate of a unique point in $\{w\} \cup G(w)$, and we extend i_1 to $i_2 : G \to J(6,3)$ in the natural way, by $i_2(\overline{y}) = \overline{i_1(y)}$ for $y \in \{w\} \cup G(w)$. Then whenever $y \notin \{w\} \cup G(w)$, $\overline{i_2(y)} = i_1(\overline{y})$.

Using quadrachotomy in $J(6,3)$, i_2 is surjective. Since $|G| = |J(6,3)|$, i_2 is also injective.

We show that $u \sim v$ iff $i_2(u) \sim i_2(v)$. There are sixteen cases, given by the comparison of u with w in (2), and the comparison of v with w. A typical case is when $u \sim w$ and $v \sim \overline{w}$. Notice that the following are equivalent:

$u \sim w$, $v \sim \overline{w}$, and $u \sim v$
$u \sim w$, $\overline{v} \sim w$, and $u \neq v$, $u \not\sim \overline{v}$, $u \neq \overline{v}$
$u \sim w$, $v \sim \overline{w}$, and $i_2(u) \neq i_2(v)$, $i_1(u) \not\sim \underline{i_1(\overline{v})}$, $i_1(u) \neq \underline{i_1(\overline{v})}$
$u \sim w$, $v \sim \overline{w}$, and $i_2(u) \neq i_2(v)$, $i_2(u) \not\sim \overline{i_2(v)}$, $i_2(u) \neq \overline{i_2(v)}$
$u \sim w$, $v \sim \overline{w}$, and $i_2(u) \sim i_2(v)$

In the last step, we used quadrachotomy in $J(6,3)$. The other cases are similar or easier. ∎

3. Finite graphs satisfying I_3 and LP_k

The Johnson graphs $J(m,n)$ are locally gridlike, and hence they do not satisfy LP_k for $k \geq 3$. To get such properties we use a construction which generalizes the structure of the Johnson graphs.

Theorem 3.3. *For every k there is a finite graph G which satisfies* I_3 *and* LP_k.

Proof. We use a probabilistic argument. For each n, consider $2n$ points,

$$\{a_1, b_1\} \cup \cdots \cup \{a_i, b_i\} \cup \cdots \cup \{a_n, b_n\}.$$

We are going to build a graph G on these points. For $1 \leq i \leq n$, we say that $\{a_i, b_i\}$ is the *ith group* of points. For each pair of distinct indices i and j, we make a graph G by either putting $a_i \sim a_j$ and $b_i \sim b_j$; or else by putting $a_i \sim b_j$ and $b_i \sim a_j$. Let $\mathcal{G}$ be the set of (labeled) graphs obtained in this way, considered as a probability space in the usual way.

The construction arranges that for all i, $d(a_i, b_i) \geq 3$. We claim that LP_k holds, almost surely in $\mathcal{G}$ as $n \to \infty$. To see this, fix some w. Consider a list $x_1, \ldots, x_p; y_1, \ldots, y_q$ of distinct neighbors of w, with $p + q \leq k$. No two points on this list can be in the same group, since the two points in each group have distance ≥ 3, and all points on the list are neighbors of w. From this, it follows that almost surely as $n \to \infty$, there is a point z as required. (The proof is a routine probabilistic argument, virtually the same as the one used to show that a random graph with edge probability $1/2$ almost surely has property P_k.)

Next, we check that almost surely, $x \sim y \sim z$ extends to a square. All three points must be in different groups. So the same probabilistic argument shows that there is a neighbor of x and z which is not a neighbor of y.

Finally, we check that every 3-chain $x \sim y \sim z$ extends isometrically to a 4-chain. Since $d(x,z) = 2$, x and z are in different groups. Let $\overline{x}$ be the element of the group of x other than x itself. Since $z \not\sim x$, the construction arranged that $z \sim \overline{x}$. As we know, $d(x,\overline{x}) \geq 3$. So $x \sim y \sim z \sim \overline{x}$ is the desired isometric extension. (Note that this fact holds for all $G \in \mathcal{G}$, not just almost surely.) ∎

4. Concluding remarks

The problem of getting finite models for I_k for $k \geq 4$ remains open. It would be nice to use a known family of graphs such as the Johnson graphs for this. We do not know if this is possible, but we do know that some of the likely candidates do not work. We consider a few families mentioned in Brouwer et al [1], along with facts from that book. The Johnson graphs themselves are locally gridlike, and no locally gridlike graph can satisfy LP_3. The Hamming graphs contain no induced pentagons; hence they will not satisfy I_4. The near polygons directly fail to have I_3. But we have not made a detailed study of the other known families of graphs, and so perhaps one or more of these holds the key to getting finite models of I_k for $k \geq 4$.

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